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Latent heats from finite-size scaling

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Abstract. A method of obtaining the latent heat at a first-order transition from finite-lattice matrix elements is discussed, and demonstrated for the case of the $(1+1)$D Potts model. The method is shown to provide smoothly convergent finite-lattice sequences. The structure of the finite-lattice Hamiltonian eigenvalues is exhibited, and a characteristic signal of the first-order transition is proposed. The finite-size scaling behaviour at the transition is also discussed.

1. Introduction

First-order transitions are difficult to treat by finite-size scaling methods (Fisher 1971, Fisher and Barber 1972, Barber 1982). The problem is that the thermodynamic functions are always analytic on a finite lattice, so that the latent heat and order parameter always vanish at the point of transition. It is, therefore, impossible to scale these quantities in the normal manner in order to estimate their (non-zero) bulk limit. Furthermore, it is sometimes extremely difficult to distinguish the first-order transition from a second-order one (Kim and Joseph 1975, Roomany and Wyld 1980, Hamer 1981).

One may try to proceed by a double limiting process: that is, by calculating the free energy and its derivatives away from the transition point, taking the lattice size to infinity (bulk limit), and then extrapolating to the transition point from either side. This should, in principle, give the correct latent heat; but it is a cumbersome and delicate procedure, since the convergence is non-uniform, and the resulting numerical accuracy is low (Hamer 1981).

A solution to this problem, due originally to Yang (1952) and Uzelac (1980, see also Uzelac and Jullien 1981) was recently discussed (Hamer 1982). The method involves consideration of the submatrix of the transfer matrix (or equivalent field theory Hamiltonian) spanned by those eigenvectors which become *degenerate* with the ground state at the transition point. In Hamer (1982), it was shown that the spontaneous magnetisation and exponent β could be calculated with excellent accuracy using this method. In the present work, we demonstrate that latent heats can be calculated using the same method, by applying it to the Potts model in $(1+1)$ dimensions. It is also shown, following Iglói and Sólyom (1982), that the structure of the finite-lattice eigenvalues provides a sensitive test to distinguish whether the transition is first or second order.

Let us first discuss the idea in general terms. In a Hamiltonian field theory framework† (Kogut 1979, Barber 1982), we may write the quantum Hamiltonian as follows

$$H = H_0(\lambda_c) + tV \tag{1}$$

where λ_c is the coupling (or temperature variable) at the transition point, and $t = \lambda - \lambda_c$ measures the distance from the transition. Suppose that the two lowest energy eigenvectors of this Hamiltonian, denoted $|1\rangle$ and $|2\rangle$, become degenerate at $t = 0$ in the bulk limit (the argument is easily generalised to the case of more than two degenerate eigenvalues). Then the ground state energy near $t = 0$ is equal to the minimum eigenvalue of the 2×2 matrix

$$\begin{pmatrix} E_0^0 + tV_{11} & tV_{12} \\ tV_{21} & E_0^0 + tV_{22} \end{pmatrix} \tag{2}$$

where E_0^0 is the unperturbed eigenvalue of states $|1\rangle$ and $|2\rangle$, and $V_{ij} \equiv \langle i|V|j\rangle$. Let μ_+ and μ_- denote the maximum and minimum eigenvalues, respectively, of the 2×2 matrix $\{V_{ij}\}$. Then it follows that the vacuum energy per site on either side of the transition point is given by

$$\begin{aligned} E_0(t)/N &\underset{N\to\infty}{\sim} (E_0^0 + t\mu_+)/N, \qquad t \to 0-, \\ E_0(t)/N &\underset{N\to\infty}{\sim} (E_0^0 + t\mu_-)/N, \qquad t \to 0+, \end{aligned} \tag{3}$$

where N is the number of sites in the lattice; and hence the latent heat per site is

$$L \equiv \lim_{t\to 0} \lim_{N\to\infty} \left\{ \frac{1}{N} \left[\frac{\partial E_0}{\partial t}(-t) - \frac{\partial E_0}{\partial t}(+t) \right] \right\} = \lim_{N\to\infty} \{(\mu_+ - \mu_-)/N\}. \tag{4}$$

Now the relation (4) depends on the exact degeneracy of the states $|1\rangle$ and $|2\rangle$, which only occurs in the bulk limit $N \to \infty$. On a finite lattice, the states are non-degenerate, and there is no discontinuity in the slope of $E_0(t)$. But the matrix eigenvalues μ_+ and μ_- are certainly non-zero, and in fact the right-hand side of equation (4) turns out to provide a smooth sequence converging to the bulk latent heat as the lattice size $N \to \infty$. Thus our finite-size scaling problem is solved.

2. Formulation

Let us apply the above method to the Q-state Potts model in $(1+1)$ dimensions. This model is known to undergo a first-order transition for $Q > 4$, and its latent heat is known exactly (Hamer 1981) from the work of Baxter (1973): so our numerical results may be compared with the exact solution. Now in order to treat the higher Q values, it is convenient to work with the equivalent eight-vertex model (Baxter 1973), whose

† To translate this argument to the ordinary Euclidean framework of statistical mechanics, one simply replaces 'quantum Hamiltonian' by (minus the) 'logarithm of the transfer matrix', 'ground-state energy by 'free energy', etc (Kogut 1979, Barber 1982).

quantum Hamiltonian takes the form (Hamer 1981)

$$H = \left(\frac{1+\lambda}{2}\right) \sum_{n=1}^{2M} \{1 - \cosh \nu(1+(-1)^n t')[(\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y) + \cosh \nu(1-\sigma_n^z \sigma_{n+1}^z)] - (-1)^n t' \sinh(2\nu)\sigma_n^z\} \tag{5}$$

where $t' = (1-\lambda)/(1+\lambda)$, $\cosh\nu = \frac{1}{2}Q^{1/2}$, and the σ_n^i are Pauli matrices acting on a two-state spin variable at each site. The time variable is continuous, while in space the system forms a one-dimensional lattice of $2M$ sites, with periodic boundary conditions. This corresponds to an M-site system in the original Potts model. Note that the number of states Q appears merely as a parameter in this version of the model.

Now the Potts model is self-dual (ignoring boundary effects), so that (Mittag and Stephen 1971, Sólyom 1981)

$$H(\lambda) = \lambda H(1/\lambda) \tag{6}$$

and the first-order transition occurs at the self-dual point, $\lambda = 1$. From equation (6) there follows the useful relation

$$\lim_{t\to 0} \lim_{M\to\infty} \left\{\frac{1}{M}\left[\frac{\partial E_0}{\partial t}(t) + \frac{\partial E_0}{\partial t}(-t)\right]\right\} = \lim_{M\to\infty} \{(\mu_+ + \mu_-)/M\} = \frac{E_0(\lambda=1)}{M} \tag{7}$$

where $t = 1-\lambda$, as in § 1. At the transition point, the following exact results are known (Yang and Yang 1966, Baxter 1973, Johnson *et al* 1973, Hamer 1981): the ground state energy in the bulk limit is

$$\lim_{M\to\infty}\left(\frac{E_0}{M}\right) = 2 - Q - Q^{1/2}(Q-4)\sum_{n=1}^{\infty} \frac{(-1)^n}{[\cosh\nu - \cosh(2n+1)\nu]}; \tag{8}$$

the latent heat is

$$L = 2\sinh 2\nu \prod_{n=1}^{\infty} \tanh^2 n\nu \tag{9}$$

$$\underset{Q\to 4+}{\sim} 4\pi Q^{1/2} \exp[-\pi^2/2(Q-4)^{1/2}]; \tag{10}$$

and the mass gap is

$$F = 2\sinh 2\nu\left(1 + 2\sum_{n=1}^{\infty} q^{n^2}\right)^{-2} \prod_{n=1}^{\infty} \tanh^4 n\nu \qquad (q = e^{-\nu}) \tag{11}$$

$$\underset{Q\to 4+}{\sim} 8\pi Q^{1/2} \exp[-\pi^2/(Q-4)^{1/2}]. \tag{12}$$

3. Results

Using the methods of Hamer and Barber (1981a), we have set up the Hamiltonian (5) on finite lattices of 2, 4, 6, . . . , 14 sites, and calculated its low-lying, translation-invariant eigenvectors and eigenvalues, for the cases $Q = 4$, 5, 6, 8 and 12. Hence the matrix $\{V_{ij}\}$ and its eigenvalues $\mu_\pm$ can be derived.

3.1. Eigenvalue structure

Consider first the structure of the finite-lattice eigenvalues near the transition point. In a most interesting recent paper, Iglói and Sólyom (1982) have shown that the eigenvalues of the original Potts model behave in a fashion illustrated in figures 1(a)–(e). The spectrum of the model breaks up into Q disjoint sectors of states, each sector labelled by the sum of all spins on the lattice, modulo Q. In the vacuum sector, the mass gap to the second lowest state behaves as shown in figure 1(a), exhibiting a dip near $\lambda = 1$ for any finite lattice size. As the lattice size M increases, this dip becomes deeper and sharper, until in the bulk limit $M \to \infty$ it approaches the curve shown in figure 1(b). Thus the mass gap at $\lambda = 1$ vanishes in this limit, producing a first-order transition as outlined above; and yet, if one takes $M \to \infty$ at fixed $\lambda \neq 1$, and then lets $\lambda \to 1$, the mass gap approaches a finite limit, corresponding to the finite bulk correlation length. Thus the Potts model manages to 'have its cake and eat it'.

In each of the $(Q-1)$ non-vacuum sectors, the mass gap to the lowest-lying state behaves as shown in figure 1(c) on a finite lattice, and develops a step-function discontinuity in the bulk limit, figure 1(d). These states become degenerate with the vacuum for $\lambda > 1$ in the bulk limit, producing the overall Q-fold degeneracy which

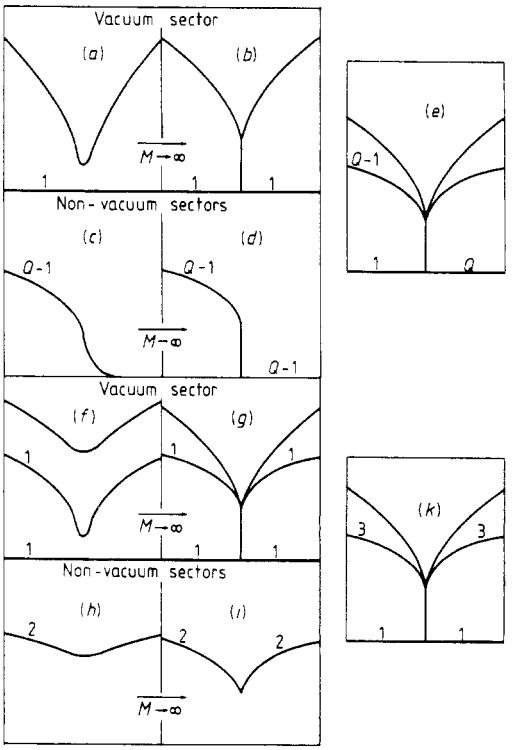

Figure 1. A schematic diagram of the finite-size scaling behaviour of low-lying Hamiltonian eigenvalues, relative to the ground-state energy. (a)–(e) describe the original Potts model (after Iglói and Sólyom 1982); (f)–(k) are for the equivalent eight-vertex model. The left-hand frame depicts the typical behaviour for finite lattice size M in each sector, while the middle frame depicts the behaviour in the bulk limit $M \to \infty$. The two right-hand frames show the overall eigenvalue pattern in the bulk limit for each model. The numbers attached to some of the eigenvalue branches are their degeneracy factors.

one expects in this region. The overall pattern of low-lying eigenvalues in the bulk limit is shown in figure 1(*e*): again, it can be seen that the mass gap remains finite as $\lambda \to 1$.

In the case of the equivalent eight-vertex model Hamiltonian, equation (5), the situation is illustrated in figures 1(*f*)–(*k*). The fundamental mechanism remains the same: the mass gap in the vacuum sector develops a spike in the bulk limit, driving the first-order transition. In the non-vacuum sectors, however, the pattern is quite different: the mass gap remains finite everywhere, and develops only a cusp singularity in the bulk limit. The overall eigenvalue pattern is shown in figure 1(*k*), and is identical to figure 1(*e*), except that the degeneracy factors are different for the various eigenvalue branches. This is just what one should expect from the operator algebra equivalence between the two models (Temperley and Lieb 1971, Baxter 1973, 1982).

To support these statements about the equivalent eight-vertex model, some numerical finite-lattice results are exhibited in figure 2, for the case $Q = 12$. Figure 2(*a*) shows some results for the mass gap in the non-vacuum sectors at lattice sizes $M \leq 7$, together with some estimates for the bulk limit, obtained by an iterated Aitken sequence extrapolation algorithm (Hamer and Barber 1981b). The gradual

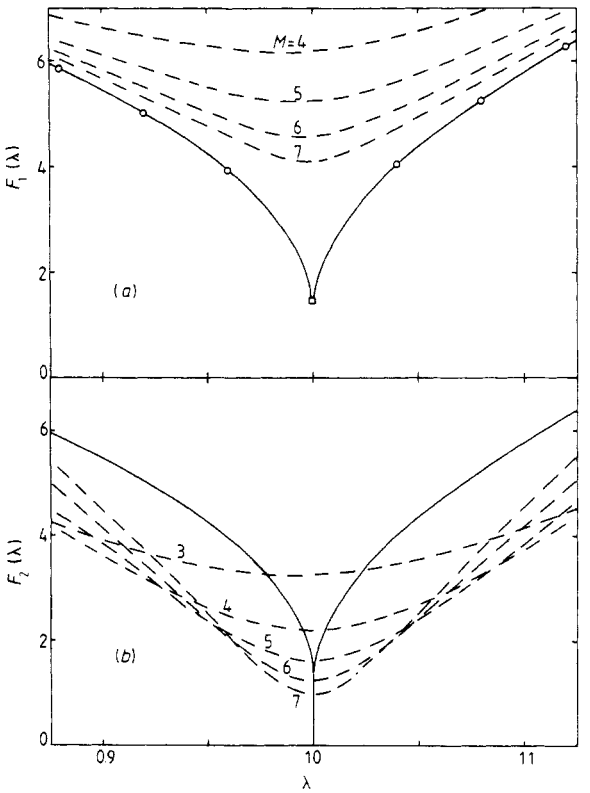

Figure 2. Mass gap values for the Potts-equivalent eight-vertex model at $Q = 12$, plotted as functions of the coupling λ. (*a*) shows the gap in the non-vacuum sectors. The broken curves are finite-lattice results, labelled by the lattice size M. The open circles are estimates of the bulk limit, obtained by sequence extrapolation from the finite-lattice results, and the open square is the exactly known result at $\lambda = 1$. (*b*) shows the gap in the vacuum sector, where the full curve represents the expected bulk limit.

development of the cusp singularity is apparent. Figure 2(*b*), by contrast, shows the mass gap in the vacuum sector, together with the expected bulk limit. Here the dip at $\lambda = 1$ is much deeper and sharper, developing into a sharp spike which runs down to touch the axis as $M \to \infty$.

3.2. Scaling properties

Let us now discuss the finite-size scaling properties of the system. If one plots the 'scaled mass-gap ratio' (Hamer and Barber 1981a) for the vacuum sector mass gap,

$$R_m(\lambda) = MF_2(\lambda, M)/(M-1)F_2(\lambda, M-1), \tag{13}$$

the result is shown in figure 3, for the case $Q = 12$. Now for an ordinary second-order critical point, one expects the mass gap to scale as $1/M$ at the critical point (Fisher 1971, Nightingale 1977, Hamer and Barber 1981a). Hence the 'pseudo-critical points' λ_M^*, defined such that $R_M(\lambda_M^*) = 1$, should converge to the bulk critical point λ_c as $M \to \infty$. By extrapolating the sequence λ_M^*, it is usually possible to obtain an accurate estimate of λ_c (Hamer and Barber 1981b). A glance at figure 3 shows that the same approach should also work for a first-order transition of the present sort.

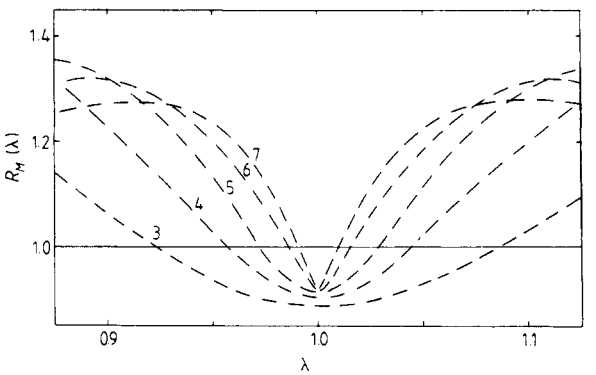

Figure 3. The scaled mass-gap ratio $R_M(\lambda)$ in the vacuum sector plotted against λ, for the case $Q = 12$. Each curve is labelled with the lattice size M. The full line is the pseudo-critical value $R_M(\lambda) = 1$.

At the point $\lambda = 1$, however, $R_M(\lambda)$ quickly approaches a limiting value which we estimate to be

$$e^{-\Gamma} \equiv \lim_{M\to\infty} \{R_M(1)\} = 0.917 \pm 0.001. \tag{14}$$

This indicates exponential† convergence in M,

$$F_2(\lambda = 1, M) \underset{M\to\infty}{\sim} \text{constant}\, e^{-\Gamma M}, \tag{15}$$

as noted already by Iglói and Sólyom (1982). This is the normal type of convergence expected when the correlation length is finite (e.g. Au-Yang and Fisher 1975).

† Or 'linear' convergence, in the paradoxical jargon of numerical analysis.

Since there has been some recent debate about finite-size scaling properties at a first-order transition (Fisher and Berker 1982, Blöte and Nightingale 1982, Barber 1982), it is perhaps worth discussing this point further. Consider again the system discussed in § 1. On a finite lattice, by Rayleigh–Schrödinger perturbation theory, we have

$$\partial E_0/\partial\lambda\,|_{\lambda=1} = \langle 1|V|1\rangle \tag{16}$$

and

$$\left.\frac{\partial^2 E_0}{\partial\lambda^2}\right|_{\lambda=1} = \sum_{n\neq 1}\frac{|\langle 1|V|n\rangle|^2}{E_1^0 - E_n^0}. \tag{17}$$

The ground-state energy E_0 is analytic on the finite lattice, but there is a peak in the specific heat per site

$$\tilde{C}_N = -N^{-1}\partial^2 E_0/\partial\lambda^2 \tag{18}$$

which develops into a delta-function singularity as $N\to\infty$, and gives rise to the latent heat discontinuity in $\partial E_0/\partial\lambda$ in that limit. Now this divergence of the specific heat is due to the asymptotic degeneracy between states $|1\rangle$ and $|2\rangle$, thus:

$$\tilde{C}_N(\lambda=1) = \frac{1}{N}\frac{|\langle 1|V|2\rangle|^2}{E_2^0 - E_1^0} + \frac{1}{N}\sum_{n\neq 1,2}\frac{|\langle 1|V|n\rangle|^2}{E_n^0 - E_1^0} \tag{19}$$

$$\leqslant \frac{1}{N}\frac{|\langle 1|V|2\rangle|^2}{E_2^0 - E_1^0} + \frac{1}{NF_{\min}}\sum_{n\neq 1,2}|\langle 1|V|n\rangle|^2 \tag{20}$$

where $F_{\min} = \min_{n\neq 1,2}\{E_n^0 - E_1^0\}$, which by assumption is non-zero. But the operator V is a sum of N finite spin operators, and so we may write

$$\tilde{C}_N(\lambda=1) < a_1 N/(E_2^0 - E_1^0) + a_2 N \tag{21}$$

where a_1, a_2 are suitably chosen constants. It follows that the finite-lattice specific heat diverges inversely with the mass gap in the vacuum sector. A similar result will apply to the susceptibility at a first-order magnetic transition.

Now for a system with finite correlation length, as mentioned above, one would normally expect exponential convergence in M†

$$F_2(\lambda=1) = E_2^0 - E_1^0 \underset{M\to\infty}{\sim} \text{constant}\, e^{-\Gamma M}, \tag{22}$$

which agrees with the behaviour demonstrated above, equation (15). Hence the specific heat will diverge exponentially at $\lambda = 1$, as observed by Blöte and Nightingale (1982).

But the (1 + 1)D field theory has a time dimension which is effectively infinite. For a fully *finite* lattice in d Euclidean dimensions, it is easy to prove (Fisher and Berker 1982) that the susceptibility and specific heat cannot diverge faster than $N = M^d$, where N is the total number of sites, and M the linear size of the system. It is, therefore, natural to expect (Imry 1980, Fisher and Berker 1982) that at a first-order transition the system will saturate this bound. Thus the scaling properties at a first-order transition depend on the geometry of the system (Barber 1982).

† Here M is the *linear* size of the lattice, i.e. the number of sites on a side, so that $N = M^d$, where d is the number of spatial dimensions of the lattice.

An alternative explanation of these scaling properties exists, in terms of the renormalisation group. Nienhuis and Nauenberg (1975) have proposed that first-order transitions are controlled by a renormalisation-group fixed point at which one eigenvalue equals d, the dimensionality of the system. Fisher and Berker (1982) have used this idea to derive the scaling behaviour outlined above for the fully finite system; and Blöte and Nightingale (1982) have used it to derive exponential scaling behaviour for a system which is infinite in one dimension.

3.3. Latent heat

Finally, let us turn to the original problem of estimating the latent heat. If the Hamiltonian (5) is decomposed as in equation (1), one obtains

$$H = H_0 + tV, \qquad t = \lambda - 1, \tag{23}$$

where

$$H_0 = \sum_{i=1}^{2M} \{1 - \cosh \nu[(\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y) + \cosh \nu(1 - \sigma_n^z \sigma_{n+1}^z)]\} \tag{24}$$

and

$$V = \tfrac{1}{2} \sum_{i=1}^{2M} \{1 - \cosh \nu(1 + (-1)^n)[(\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y) + \cosh \nu(1 - \sigma_n^z \sigma_{n+1}^z)] - (-1)^n \sinh(2\nu)\sigma_n^z\}. \tag{25}$$

The two lowest eigenvectors of H_0 in the vacuum sector can then be calculated by standard methods (Hamer and Barber 1981a). A basis of eigenstates of σ_n^z was used to generate a matrix representation of H_0 for each lattice size M; and the low-lying eigenvectors of this matrix were computed using a conjugate gradient method. The 2×2 matrix $\{V_{ij}\}$ and its eigenvalues could then be calculated, to an expected accuracy of nine significant figures.

The quantity $L_M = (\mu_+ - \mu_-)/M$ provides an estimate of the latent heat for each lattice size M, from equation (4). It is found that these estimates form a smoothly convergent sequence, approaching the bulk latent heat as $M \to \infty$. The convergence of this sequence can be accelerated using the VBS algorithm (Vanden Broeck and Schwartz 1979, Hamer and Barber 1981b), which depends on a parameter α. Table 1 shows an example, for the case $Q = 12$. Here the value $\alpha = 0$ was found to give

Table 1. A sequence extrapolation for the latent heat of the 12-state Potts model, using an iterated Aitken algorithm. The left-hand column lists the quantity $(\mu_+ - \mu_-)/M$ (see text) for lattice sizes $M = 1, 2, \ldots, 8$.

M				
1	12.000 000 00			
2	8.657 386 37	7.624 531 08		
3	7.868 342 67	6.864 193 22	6.425 462 35	
4	7.426 495 23	6.585 990 91	6.323 296 04	6.195 759 18
5	7.136 890 86	6.450 877 77	6.266 570 79	6.240 371 23
6	6.933 253 27	6.372 916 82	6.248 648 81	
7	6.783 895 36	6.325 010 47		
8	6.671 213 24			

the best convergence: then the VBS algorithm is equivalent to an iterated Aitken algorithm. This is appropriate for a 'linearly' convergent sequence ($L_M - L_\infty \sim c_1 \exp(-c_2 M)$), which is the expected finite-lattice behaviour except in the neighbourhood of a second-order critical point. By these means, reasonably accurate estimates of the bulk latent heat can be obtained. As a check, the sequence $(\mu_+ + \mu_-)/M$ was treated by the same methods: this should converge to E_0/M, from equation (7).

The results are shown in table 2, where the numerical estimates obtained by sequence extrapolation are compared with the exactly known values. It can be seen that the estimates of E_0/M attain excellent accuracy, down to the fourth decimal place. The latent heat estimates are not so precise. It appears that the quantity $(\mu_+ - \mu_-)/M$ converges with an exponent something like half that of the quantity $(\mu_+ + \mu_-)/M$. It is also apparent that the accuracy decreases as Q approaches 4, and the effective scaling behaviour gradually switches over from 'linear' convergence to 'logarithmic' convergence ($L_M - L_\infty \sim c_1 M^{-c_2}$). Correspondingly, the favoured α value switches from 0 to -1 (Hamer and Barber 1981b). When Q is precisely *equal* to 4 (the critical point), the convergence is purely logarithmic, and the VBS algorithm once again gives excellent accuracy.

Table 2. Numerical results are shown for the ground-state energy per site (E_0/M) and latent heat (L) of Q-state Potts models at $\lambda = 1$, obtained by the method described in the text. The known exact values are also shown for comparison. The column α lists the parameter value used in the VBS sequence extrapolations.

		E_0/M		L	
Q	α	Exact	Numerical estimate	Exact	Numerical estimate
4	−1	−3.545 18	−3.5453 ± 0.0005	0	(0 ± 1)E−4
5	−1	−4.622 54	−4.6227 ± 0.0005	0.173 18	0.25 ± 0.2
6	−1	−5.679 22	−5.6795 ± 0.0005	0.779 60	0.77 ± 0.2
8	0	−7.754 96	−7.7554 ± 0.0004	2.453 6	2.3 ± 0.1
12	0	−11.834 70	−11.8346 ± 0.002	6.242 0	6.24 ± 0.04

The convergence exponent at $Q = 4$, defined by

$$L_M \underset{M\to\infty}{\sim} \text{constant}\, M^{-\sigma}, \tag{26}$$

may be estimated from the numerical results, and is found to be

$$\sigma = 2.003 \pm 0.01 \tag{27}$$

using the methods of Hamer and Barber (1981b). Now from the usual finite-size scaling hypothesis (Fisher 1971, Fisher and Barber 1972, Hamer and Barber 1981a), this should correspond to the bulk behaviour,

$$L \underset{Q\to 4+}{\sim} F^\sigma, \tag{28}$$

where F is the mass gap, or inverse correlation length. But from equations (10) and (12), one finds that the expected value of σ is $\frac{1}{2}$. Thus finite-size scaling appears to *break down* in this instance. The whole question of finite-size scaling in the vicinity of a Kosterlitz–Thouless transition merits closer investigation (Barber 1982).

4. Discussion and summary

In this paper, we have explored a new and direct method for evaluating the latent heat at a first-order transition using finite-size scaling. The method involves the calculation of a submatrix spanned by the degenerate ground-state eigenvectors at the transition point. It has been shown to provide a smoothly convergent sequence of latent heat estimates for finite M, which may be reliably extrapolated to the bulk limit.

Some additional evidence has also been found for the remarkable picture of Iglói and Sólyom (1982), showing the structure of the finite-lattice Hamiltonian eigenvalues at a thermal first-order transition. (The details of this structure were presented in § 3.1.) Here, let us merely remark on the characteristic differences between a first- and second-order transition. At a second-order transition, a whole *band* of eigenvalues scale down together like $1/M$ towards the ground-state energy, led in the typical case by the mass gap in the *non*-vacuum sector. In the case of a thermal first-order transition, it is only a *finite* set of eigenvalues in the vacuum sector that dip sharply down to become degenerate with the ground state in the bulk limit. Thus the spectrum will not show the smooth scaling behaviour characteristic of a second-order transition: and in particular, we may expect a *crossing* of energy levels as the mass gap in the vacuum sector drops below that in the non-vacuum sector. In the Potts-equivalent eight-vertex model†, for instance, the dividing line is the case $Q = 4$, where the two mass gaps are in fact identical for all λ. For $Q > 4$, the vacuum mass gap is always smaller. We expect that this characteristic crossing of levels should provide a useful signal of thermal first-order transitions in more general models.

Some discussion was also given of the finite-size scaling behaviour expected near a first-order transition. It was argued that one may normally expect exponential ('linear') convergence to the bulk limit, due to the finite correlation length at the transition. From this point of view, the 'logarithmic' scaling behaviour found in the fully finite system (i.e., finite in all dimensions) is anomalous, being imposed by the special *constraints* on that system (Imry 1980, Fisher and Berker 1982). We thus argue against the idea that finite-size scaling behaviour at a first-order transition is governed by a renormalisation-group fixed point.

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† In the original Potts model, we would expect a much more dramatic crossing of energy levels near the transition point for $Q > 4$, *provided* periodic boundary conditions are used (Iglói and Sólyom 1982).

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